Max-Plus Control Design for Temporal Constraints Meeting in Timed Event Graphs

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***Abstract*—The aim of the presented work is the control of Timed Event Graph to meet tight temporal constraints. The problem of temporal constraints is formulated in terms of control of linear Max-Plus models. First, the synthesis of a control law that ensures the satisfaction of a single constraint for a single input system is presented. Then, the single input multi-constraints problem is tackled and finally, the method is extended to the multi-inputs, multi-constraints problem. The proposed method is illustrated on the example of a simple production process.**

***Index Terms*—Discrete event systems (DES), feedback control, max-plus algebra, temporal constraints, timed event graph (TEG).**

I. Introduction

Many manufacturing systems are subject to tight time constraints in their production process. Such constrained plants can be found in the semiconductor industry [17], or in the automotive industry [21]. The aim of the presented work is to design a method to manage such a plant, so the strict time constraints are satisfied.

Production control and manufacturing plant problems are generally considered as Discrete Event Systems (DES) control problems. Petri Nets have been extensively used [9], [23] to deal with DES. Event graphs (a subclass of Petri nets) are often used, when the possible choices, for instance in the routing, are resolved with the help of the production control. We consider in the sequel a class of deterministic controlled processes modeled with Timed Event Graphs (TEG), that permit to take time constraints explicitely into account.

The problem tackled here could be formulated as a verification of some temporal conditions, see for instance [6], [10], [13]. In the present contribution, the verification of time constraints is formulated in terms of a control problem, assuming that some inputs of the process can be controlled (it is generally the case). As is well-known from [5], TEG give rise to Max-Plus algebraic models, which are linear over the dioid $\mathbb{R}_{\max}$, and we make use of this linear framework. The plant behavior is modeled with Max-Plus equations, and the temporal constraints are represented with inequalities, also linear in the Max-Plus algebra. We propose a method for the synthesis of control law that permits to meet a given set of time constraints. The resulting control law itself is finally defined as a Max-Plus linear difference equation, involving a finite number of delays. This control law is causal, and can be implemented on-line, from the knowledge of the system state. Such an equation corresponds to a feedback that is also a TEG.

The control approach that we propose is quite different from that considered within the so-called supervisory control framework of timed discrete event systems ([15], [22]). Time is explicitly taken into account in the proposed approach. TEGs and dioid framework have been used in the literature to treat a number of control problems. In [5] is calculated the latest control that permits to match a given target. The computation is based on residuation theory, and in terms of manufacturing processes, this control is interpreted in terms of a just-in-time production strategy. The residuation theory was further used by Cottenceau *et al.* [11], to compute a feedback that delays as far as possible the input of the system without altering the input-output transfer relationship. It has also been used for time-varying systems in [18], and for uncertain systems with uncertain targets in [19]. Houssin *et al.* in [14] propose a control strategy for the system to meet a set of general constraints, that includes time constraints. In the case that is considered here, the latest control (just-in-time) leads to a trivial behavior, if no more constraint is specified. The interpretation for a manufacturing system is that when the production is stopped, no part violates the time constraint, but this is not a pertinent solution to our problem. One seeks a control that is as permissive as possible, which means as early as possible. Unfortunately, there is no optimal control in that sense, since the usual method for the synthesis of control strategies for Max-Plus systems, that are based on residuation theory, does not apply in our case.

An interesting formulation of our synthesis problem can be given in terms of controlled invariance. Katz [16] has shown that the satisfaction of linear inequality constraints defined in terms of a matrix E can be stated in terms of (A, B)-invariance of a semimodule included into the image of the Kleene star E^* (see Section II-A for the definition of the Kleene star operation). The control design hence comes down to the computation of the maximal (A, B)-invariant semimodule included into the image of E^*, as illustrated in Section VI of [16]. Unfortunately, this computation is difficult, and no general procedure is known for this computation. It may happen that this semimodule is not finitely generated, or that the (A, B)-invariant algorithm does not stabilize, because the dioid $\mathbb{R}_{\max}$ is not artinian. We consider the case where not all components of the state are constrained, hence some entries of E^* are not finite. As a consequence the condition proposed in [16] for the maximal (A, B)-invariant semimodule to be finitely generated is not satisfied, and we cannot use this formulation.

A first attempt to control TEG under strict temporal constraints has been presented in [4]. This initial approach was developed in the Min-Plus algebra, under the assumption that all delays of the considered graph are integers. In the present contribution, this condition is not required, i.e., we consider a TEG with delays that may be real numbers.

The technical note is organized as follows. In Section II, some backgrounds on Max-Plus algebra and TEGs are recalled. The problem of finding a causal control law that satisfies critical time constraints is formulated in Section III. In Section IV, we first introduce a sufficient condition for the existence of a feedback under a single input and a single constraint. The method is constructive, and a suitable feedback is proposed, provided that the condition is satisfied. Then, the method is extended to the case of several temporal constraints. Finally, the general case, i.e., the characterization of a causal control law that guarantees several temporal constraints of a multi-input dynamic system modeled by a TEG is proposed in Section V. Section VI is devoted to an illustrative example. Concluding remarks are presented in Section VII.

II. Backgrounds

A. Max-Plus Algebra

A monoid is a set, say $\mathcal{D}$, endowed with an internal law, noted $\oplus$, which is associative and has a neutral element, denoted ε. A semiring is a commutative monoid endowed with a second internal law, denoted $\otimes$, which is associative, distributive with respect to the first law $\oplus$, has a neutral element, denoted e, and admits ε as absorbing element: $\forall a \in \mathcal{D}, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$. A dioid is a semiring with an idempotent internal law: $\forall a \in \mathcal{D}, a \oplus a = a$. The dioid is said to be commutative

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if the second law $\otimes$ is commutative. Max-Plus algebra is defined as $(\mathbb{R}\cup\{-\infty\}, \max, +)$. This semi-ring, denoted $\mathbb{R}_{\max}$, is a commutative dioid, the law $\oplus$ is the operation $\max$ with the neutral element $\varepsilon = -\infty$, and the second law $\otimes$ is the usual addition, with neutral element $e = 0$. For $n \in \mathbb{N}$ and $V, W \in \mathbb{R}_{\max}^{n\times n}$, by $V \oplus W$ we denote the matrix with components $(V \oplus W)_{ij} = \max(V_{ij}, W_{ij})$. Given $p, q \in \mathbb{N}$, and matrices $A \in \mathbb{R}_{\max}^{p\times n}$ and $B \in \mathbb{R}_{\max}^{n\times q}$, the symbol $A \otimes B$ (or just $A.B$) will denote the result of matrix multiplication defined by the formula

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} (A_{ik} \otimes B_{kj}) = \max_k (A_{ik} + B_{kj}).$$

The Kleene star of a square matrix $M \in \mathbb{R}_{\max}^{n\times n}$, written M^* is defined as $M^* = \bigoplus_{i\in\mathbb{N}} M^i$, where M^0 equals the unit matrix, with entries equal to e on the diagonal, and ε elsewhere. Let us recall that for $x, v \in \mathbb{R}_{\max}^{n}$ the minimal solution of both the inequality, $x \geq M.x \oplus v$, and the equality, $x = M \otimes x \oplus v$, is $x = M^* \otimes v$ [5]. We denote by e^i the vector of $\mathbb{R}_{\max}^{n}$ with its ith entry equal to e, and the other entries equal to ε.

The following lemma and theorem will be useful to determine the existence of a solution to the problem of multivariable control tackled in this article. Lemma 1 concern a specific case of $A \otimes x \leq B \otimes x$, see [1] and [7], [8]. We provide it with its proof, that is simple, since it gives some hints for the proof of Theorem 1, which is more complicated, and for the sake of completion, because we use it in the proof of Lemma 2, in Section V.

Lemma 1: For any two row vectors u, v in $\mathbb{R}_{\max}^{1\times n}$, there exists a non trivial solution to inequality $u \otimes x \leq v \otimes x$, if and only if there is $k \in \bar{n}$, such that $u_k \leq v_k$.

Proof: The vector $x = \varepsilon$ is always a solution, called the trivial solution. A solution x is called non trivial if there exists an index k so that $x_k \neq \varepsilon$. Under this condition, we can choose k so that also $v \otimes x = v_k + x_k$. Therefore we have $u_k + x_k \leq u \otimes x \leq v \otimes x = v_k + x_k$, which, since $u_k \neq \varepsilon$, shows the necessity of the statement. Conversely, under the stated condition, one can choose $x_k = e$, and $x_i = \varepsilon$, for $i \neq k$, which defines a non trivial solution. ■

Theorem 1 gives explicitly the set of solutions to $A \otimes x \leq B \otimes x$; details are given in [1] and proof in [2].

Theorem 1: Let $\mathcal{C} \subset \mathbb{R}_{\max}^{n}$ be defined as the set $\{x \in \mathbb{R}_{\max}^{n} \mid Ax \leq Bx\}$, where $A, B \in \mathbb{R}_{\max}^{q\times n}$ (with $q > 0$). Let $G_0, G_1, \ldots, G_q$ denote the sequence of finite subsets of $\mathbb{R}_{\max}^{n}$ defined as follows:

$$\begin{cases} G_0 = & e^i, i \in \overline{n} \\ G_\ell = & \{g \in G_{\ell-1} \mid A_\ell \otimes g \leq B_\ell \otimes g\} \\ & \cup\{(A_\ell \otimes h) \otimes g \oplus (B_\ell \otimes g) \otimes h \mid g, h \in G_{\ell-1}, \\ & A_\ell \otimes g \leq B_\ell \otimes g, \text{ and } A_\ell \otimes h > B_\ell \otimes h\} \end{cases}$$

for all $\ell \in \overline{q}$, where A_ℓ and B_ℓ are the ℓ-th rows of matrices A and B. Then $\mathcal{C}$ is generated by the finite set G_q.

B. TEGs, Linear Max-Plus Models

An event graph is an ordinary Petri net where each place has exactly one upstream transition and one downstream transition. A TEG is obtained by associating delays to places or to transitions of a given event graph. In our case, delays are associated to places. Let $t_1, \ldots, t_n$ be transitions having at least one upstream place, and let $t_1^u, \ldots, t_q^u$ be transitions having no upstream place and assumed to be controllable. If it exists, the place linking t_j to t_i, unique by hypothesis, is denoted by p_{ij}, its corresponding delay is denoted τ_{ij} and its marking, i. e. the number of tokens within this place, is denoted by m_{ij}.

Transition t_j is controllable if there exists a path from transition t_s^u to transition t_j. Such a path α is an alternating sequence of transitions and places, of the form $[t_s^u, p_{k_1 s}, t_{k_1}, \ldots, t_{k_l}, p_{jk_l}, t_j]$. By m_α we denote

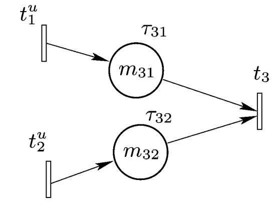

Fig. 1. TEG example.

the sum of markings along path α, such as: $m_\alpha = m_{k_1 s} \otimes \cdots \otimes m_{jk_l} = m_{k_1 s} + \cdots + m_{jk_l}$.

To represent the dynamic behavior of TEGs in Max-Plus algebra, a firing time $u_s(k)$ of the kth occurrence is associated with each transition t_s^u and a firing time $\theta_i(k)$ of the kth occurrence is associated with each transitions t_i.

Example 1: The dynamic behavior of the TEG in Fig. 1 is characterized by the following equation:

$$\theta_3(k) = \max(\tau_{31} + u_1(k - m_{31}), \tau_{32} + u_2(k - m_{32}))$$

which, in Max-Plus algebra appears to be linear

$$\theta_3(k) = \tau_{31} \otimes u_1(k - m_{31}) \oplus \tau_{32} \otimes u_2(k - m_{32})$$

The dynamic behavior of a TEG can be expressed by means of a linear equation in Max-Plus algebra as follows:

$$\theta(k) = \bigoplus_{m\geq 0} [A_m \otimes \theta(k - m) \oplus B_m \otimes u(k - m)] \quad (1)$$

where the components of vector $\theta(k)$ are the firing times of the n transitions t_i, the components of $u(k)$ are the firing times of source transitions t_s^u. Matrix A_m belongs to dioid $\mathbb{R}_{\max}^{n\times n}$, its entry $A_{m_{ij}}$ equals to τ_{ij} if there exists a place p_{ij} containing m_{ij} tokens, otherwise $A_{m_{ij}}$ equals to ε. Similarly, the entries of matrices $B_m \in \mathbb{R}_{\max}^{n\times q}$ correspond to the delays of places following source transitions. It is worth replacing (1) by the following explicit equation:

$$\theta(k) = \bigoplus_{m>0} \left[A_0^* \otimes A_m \otimes \theta(k - m) \oplus A_0^* \otimes B_m \otimes u(k - m) \right] \quad (2)$$

where A_0^* is the Kleene star of A_0, previously defined (see [5] for more details). Entries of A_0 are the delays associated to places without any token, therefore for a live event graph, entries of A_0^* are such that $(A_0^*)_{ij} < +\infty$.

Analogously to the case of usual linear systems, the explicit (2) can be brought into a state space form. In order to obtain a state space model, one first expands all the places with a marking $m > 1$ into m places with a marking equal to 1. Hence $(m - 1)$ intermediate transitions are added. Then the resulting extended state vector $x(k) \in \mathbb{R}_{\max}^{N}$ is obtained, $N = n + (m - 1)$. The dynamic behavior of the expanded TEG is then described by an equation of the form, $x(k) = \hat{A}_0 \otimes x(k) \oplus \hat{A}_1 \otimes x(k-1) \oplus \hat{B} \otimes u(k)$, with $\hat{A}_0, \hat{A}_1 \in \mathbb{R}_{\max}^{N\times N}$ and $\hat{B} \in \mathbb{R}_{\max}^{N\times q}$ which can be rewritten into the following explicit form, for $k \geq 1$:

$$x(k) = A \otimes x(k - 1) \oplus B \otimes u(k) \quad (3)$$

where $A = \hat{A}_0^* \otimes \hat{A}_1$, $B = \hat{A}_0^* \otimes \hat{B}$, and initial conditions for $\ell = 1$ to N are $x_\ell(0) = \varepsilon$, the canonical initial conditions [5]. These formulations permit to point out that the behavior of a controlled TEG is fully

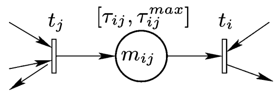

Fig. 2. Temporal constraint.

determined by the input $u(k)$, its state (3) and canonical initial conditions. Hence, the following formulation should be used:

$$x(k) = A^{\varphi} \otimes x(k-\varphi) \oplus \left[\bigoplus_{k'=0}^{\varphi-1} A^{k'} \otimes B \otimes u(k-k')\right] \quad (4)$$

for each integer $1 \leq \varphi \leq k$. Let us assume that the input $u(k)$ is actually a control input, which can be arbitrarily assigned.

III. Problem of Temporal Constraints

Strict time constraints are frequent in industry. We took inspiration from the examples reported in [3] and [17], that respectively concern rubber transformation for the automotive industry, and semiconductor manufacturing. One can for instance consider the example of a production process with a furnace for realizing a thermal treatment. The duration of any treatment in the furnace is fixed, or defined by a time interval. The aim is to control the system to meet this time constraint. A TEG already takes into account a delay on each place that corresponds to a minimal holding time. Nevertheless, in order to take into account the maximal duration, one has to express an additional constraint that the system should meet. The sojourn time of tokens in place p_{ij} may be greater or equal to τ_{ij}. In the presented work, a maximum sojourn time, noted $\tau_{ij}^{\max}$ must also be respected. Hence, an interval of time $[\tau_{ij}, \tau_{ij}^{\max}]$ is associated to place p_{ij} subject to a strict time constraint (see Fig. 2). This additional temporal constraint is expressed, for $k \geq m_{ij}$, through the following inequality:

$$x_i(k) \leq \tau_{ij}^{\max} \otimes x_j(k - m_{ij}). \quad (5)$$

IV. Single Control

In this section, TEGs with a single source transition t^u are considered. Let us assume that the control law $u(k)$ applied to this transition t^u is given by a feedback $u(k) = F \otimes x(k-1)$, for $k \geq 2$ and $u(1) = e$. F is a row vector with N entries which are either $F_r = \varepsilon$ or $F_r \geq 0$, they correspond to the delay associated to control places that are to be determined to satisfy the constraints. $F_r = \varepsilon$ means the absence of control place between t_r and t^u. It also means that the control u does not depend on x_r.

A. Single Constraint

Let us consider a TEG modelled by the linear Max-Plus (3) and subject to a single temporal constraint (5) on place p_{ij}. Let us assume that transition t_j is controllable, i.e., there exists a path α from t^u to t_j where m_α is the cumulated marking along this path.

Theorem 2: Let be given the system (3), that defines the behavior of a TEG subject to a single constraint of the form (5) on place p_{ij}. We assume that the cumulated marking along the path from the control input to place p_{ij} is zero, $m_\alpha = 0$, as well as the initial marking in the constrained place, $m_{ij} = 0$. Then, the temporal constraint is satisfied applying a feedback of the form $u(k) = F \otimes x(k-1)$, if the following condition is satisfied:

$$B_i \leq \tau_{ij}^{\max} \otimes B_j.$$

In addition, the following set of inequalities defines a suitable feedback, that guarantees that the temporal constraint is satisfied:

$$F_r \otimes B_j \otimes \tau_{ij}^{\max} \geq A_{ir}, \text{ for } r = 1 \text{ to } N.$$

Proof: First remark that, by assumption, there exists a path from the control transition t^u to transition t_j, then $B_j \neq \varepsilon$. Hence, a finite feedback F will always exist under the conditions of Theorem 2. According to (4), the ith component of the vector $x(k)$ is given by the following explicit expression:

$$x_i(k) = \bigoplus_{r=1}^{N} A_{ir}^{\varphi} \otimes x_r(k-\varphi) \oplus \left[\bigoplus_{k'=0}^{\varphi-1} (A^{k'} \otimes B)_i \otimes u(k-k')\right] \quad (6)$$

for every integer $\varphi \geq 1$. Then, considering only the contribution of the control input, e.g., the relation between transitions t^u and t_j, we obtain that the jth state vector component $x_j(k)$ satisfies (7)

$$(A^{m_\alpha} \otimes B)_j \otimes u(k - m_\alpha) \leq x_j(k). \quad (7)$$

Taking (6) into account, we obtain that the constraint (5) is satisfied if the two following inequalities hold:

$$\bigoplus_{r=1}^{N} A_{ir}^{\varphi} \otimes x_r(k-\varphi) \leq \tau_{ij}^{\max} \otimes x_j(k - m_{ij}), \quad (8)$$

$$\bigoplus_{k'=0}^{\varphi-1} (A^{k'} \otimes B)_i \otimes u(k-k') \leq \tau_{ij}^{\max} \otimes x_j(k - m_{ij}). \quad (9)$$

Further, taking (7) into account, we obtain that inequalities (8) and (9) are satisfied if the following inequalities hold:

$$\bigoplus_{r=1}^{N} A_{ir}^{\varphi} \otimes x_r(k-\varphi) \leq \tau_{ij}^{\max} \otimes (A^{m_\alpha} \otimes B)_j \otimes u(k - m_\alpha - m_{ij}), \quad (10)$$

$$\bigoplus_{k'=0}^{\varphi-1} (A^{k'} \otimes B)_i \otimes u(k-k') \leq \tau_{ij}^{\max} \otimes (A^{m_\alpha} \otimes B)_j \otimes u(k - m_\alpha - m_{ij}). \quad (11)$$

Inequality (10) is satisfied if one can determine a feedback F such that $u(k) \geq \bigoplus_{r=1}^{N} \left[A_{ir}^{\varphi} - (A^{m_\alpha} \otimes B)_j - \tau_{ij}^{\max}\right] \otimes x_r(k-1)$ with $\varphi = m_{ij} + m_\alpha + 1$. The former expression defines suitable causal control laws $u(k)$ with the form $u(k) = F \otimes x(k-1)$ if condition (12) holds for $r = 1$ to N

$$F_r \otimes (A^{m_\alpha} \otimes B)_j \otimes \tau_{ij}^{\max} \geq A_{ir}^{\varphi}. \quad (12)$$

Condition (11) holds true if inequalities (13) and (14) are satisfied for $k' = 0$ to $\varphi - 1$

$$(A^{k'} \otimes B)_i \leq \tau_{ij}^{\max} \otimes (A^{m_\alpha} \otimes B)_j, \quad (13)$$

$$u(k-k') \leq u(k - m_\alpha - m_{ij}). \quad (14)$$

As the function $u(k)$ is non-decreasing, the inequality (14) is satisfied if the inequality $k' \leq m_\alpha + m_{ij}$ is satisfied for $k' = 0$ to $\varphi - 1$. By assumption $m_\alpha = m_{ij} = 0$ this inequality is always true. Again with $m_\alpha = m_{ij} = 0$, we have $\varphi = 1$ and hence condition (12) reduces to $F_r \otimes B_j \otimes \tau_{ij}^{\max} \geq A_{ir}$, for $r = 1$ to N, condition (13) reduces to $B_i \leq \tau_{ij}^{\max} \otimes B_j$. ■

Remark 1: The hypotheses that $m_\alpha = m_{ij} = 0$, are met in general for production plants. It means that at initial state, there is no product in process in the plant, which is not restrictive in practice.

Remark 2: A suitable feedback F is found choosing $F_r = \varepsilon$ for $A_{ir} = \varepsilon$ and $F_r = \max(0, A_{ir} - B_j - \tau_{ij}^{\max})$ for $A_{ir} \neq \varepsilon$.

B. Multiple Constraints

We consider now the case of a TEG having one source transition, which is controllable, with Z constrained places. These places are denoted p_z, for $z = 1$ to Z. For each constrained place p_z, let m_z, τ_z and $\tau_z^{\max}$ respectively denote the initial marking, the minimal and maximal delays. Further, let t_z and $t_{z'}$ respectively denote the input and output transitions of place p_z, let $x_z(k)$ and $x_{z'}(k)$ denote the corresponding firing dates, and let m_{α_z} denotes the cumulated marking along path α_z going from source transition t^u to t_z. The temporal constraints are now expressed by the inequalities

$$x_{z'}(k) \leq \tau_z^{\max} \otimes x_z(k - m_z) \tag{15}$$

for $z = 1$ to Z. We denote by $u^z(k)$ the control law calculated as in the previous section to satisfy the zth temporal constraint. The following theorem defines a causal feedback, which ensures that all Z temporal constraints are satisfied.

Theorem 3: Let be given the system (3), subject to Z time constraints of the form (15), where $m_z = m_{\alpha_z} = 0$, for $z = 1$ to Z. Then, there exists a causal control law of the form $u(k) = \bigoplus_{z=1}^{Z} u^z(k)$, with $u^z(k) = F_z \otimes x(k-1)$, that ensures the satisfaction of the Z temporal constraints, if the following condition is satisfied:

$$B_{z'} \leq \tau_z^{\max} \otimes B_z$$

A suitable feedback has to satisfy the following inequality:

$$F_{zr} \otimes B_z \otimes \tau_z^{\max} \geq A_{z'r}, \text{ for } r = 1 \text{ to } N.$$

Proof: From Theorem 2, one observes that the conditions of Theorem 3 are sufficient for the feedback $u^z(k)$ to satisfy the zth temporal constraint. Since the inequality $\bigoplus_{\ell=1}^{Z} u^\ell(k) \geq u^z(k)$, is true for $z = 1$ to Z, it is clear that $u(k) = \bigoplus_{\ell=1}^{Z} u^\ell(k)$ fulfills all the Z temporal constraints, which ends the proof. ∎

V. Multivariable Control

Let us consider in this section a TEG with q source transitions, $q \geq 1$. The behavior of such a graph is again represented by a linear Max-Plus system (3), the control law being a vector u of q components. Let us first suppose that a single place p_{ij} is subject to a temporal constraint of the form (5). The problem is to design a control law $u(k) \in \mathbb{R}_{\max}^q$, with $q \geq 1$, to satisfy the constraint (5). The components of $u(k)$ are denoted by $u_\ell(k)$ for $\ell = 1$ to q. We denote by m_{α_ℓ} the cumulated markings along path α_ℓ from a controllable transition t_ℓ^u to transition t_j.

Lemma 2: Let a TEG with q source transitions $(q \geq 1)$ be subject to a single temporal constraint of the form (5) on place p_{ij}. Then, there exits a feedback of the form $u(k) = F \otimes x(k-1)$ to guarantee the satisfaction of the temporal constraint (5) if there exists an index ℓ such that $\tau_{ij}^{\max} \otimes B_{j\ell} \geq B_{i\ell}$, with $m_{\alpha_\ell} = 0$.

A suitable feedback is obtained with

$$\tau_{ij}^{\max} \otimes \bigoplus_{\ell=1}^{q} (B_{j\ell} \otimes F_{\ell r}) \geq A_{ir} \oplus \bigoplus_{\ell=1}^{q} (B_{i\ell} \otimes F_{\ell r})$$

for $r = 1$ to N.

Proof: If all $B_{j\ell} = \varepsilon$, no solution can be found. This means that no path exists from any controllable transition t_ℓ^u to transition t_j. If there exist q controllable transitions, then the jth state vector component $x_j(k)$ satisfies the inequality

$$\bigoplus_{l=1}^{q} (A^{m_{\alpha_\ell}} \otimes B)_{j\ell} \otimes u_\ell(k - m_{\alpha_\ell}) \leq x_j(k). \tag{16}$$

According to (4), the ith component of vector $x(k)$ is given by the following explicit expression:

$$x_i(k) = \bigoplus_{r=1}^{N} A_{ir}^{\varphi} \otimes x_r(k-\varphi) \oplus \bigoplus_{\ell=1}^{q} \left[\bigoplus_{k'=0}^{\varphi-1} (A^{k'} \otimes B)_{i\ell} \otimes u_\ell(k-k') \right] \tag{17}$$

for every integer $\varphi \geq 1$. Taking (17) into account, we obtain that the constraint (5) is satisfied if the two following inequalities hold:

$$\bigoplus_{r=1}^{N} A_{ir}^{\varphi} \otimes x_r(k-\varphi) \leq \tau_{ij}^{\max} \otimes x_j(k - m_{ij}), \tag{18}$$

$$\bigoplus_{\ell=1}^{q} \bigoplus_{k'=0}^{\varphi-1} (A^{k'} \otimes B)_{i\ell} \otimes u_\ell(k-k') \leq \tau_{ij}^{\max} \otimes x_j(k - m_{ij}). \tag{19}$$

Further, taking (16) into account, inequalities (18) and (19) are satisfied if the following inequalities hold:

$$\bigoplus_{r=1}^{N} A_{ir}^{\varphi} \otimes x_r(k-\varphi) \leq \bigoplus_{\ell=1}^{q} \tau_{ij}^{\max} \otimes (A^{m_{\alpha_\ell}} \otimes B)_{j\ell} \otimes u_\ell(k - m_{\alpha_\ell} - m_{ij}), \tag{20}$$

$$\bigoplus_{\ell=1}^{q} \bigoplus_{k'=0}^{\varphi-1} (A^{k'} \otimes B)_{i\ell} \otimes u_\ell(k-k') \leq \bigoplus_{\ell=1}^{q} \tau_{ij}^{\max} \otimes (A^{m_{\alpha_\ell}} \otimes B)_{j\ell} \otimes u_\ell(k - m_{\alpha_\ell} - m_{ij}). \tag{21}$$

By assumption, $u = F \otimes x(k-1)$ and $m_{ij} = 0$. If there exists a path α_s with $m_{\alpha_s} = 0$, then with $\varphi = m_{ij} + m_\alpha + 1$, one can always satisfy inequality (20) by choosing a feedback F such that $[B_{js} \otimes F_{sr}] \otimes \tau_{ij}^{\max} \geq A_{ir}$, for $r = 1$ to N. On the other hand, inequality (21) is satisfied if $\bigoplus_{\ell=1}^{q} B_{i\ell} \otimes F_{\ell r} \otimes x_r(k) \leq \bigoplus_{\ell=1}^{q} \tau_{ij}^{\max} \otimes B_{j\ell} \otimes F_{\ell r} \otimes x_r(k)$ for $r = 1$ to N, hence if $\bigoplus_{\ell=1}^{q} B_{i\ell} \otimes F_{\ell r} \leq \bigoplus_{\ell=1}^{q} \tau_{ij}^{\max} \otimes B_{jl} \otimes F_{\ell r}$ for $r = 1$ to N. As stated by Lemma 1, if $\exists \ell \in \overline{q} \mid \tau_{ij}^{\max} \otimes B_{j\ell} \geq B_{i\ell}$, then there exists a solution vector F_r for each column r. ∎

Let us now consider a TEG with Z temporal constraints. If there exists a path from a controllable transition t_s^u to transition t_z for each Z constraints, with $m_{\alpha_z} = m_z = 0$, then one can synthesize a control law that satisfies all Z temporal constraints using Theorem 4.

Theorem 4: Let a TEG with q source transitions $(q \geq 1)$ be the subject to Z additive temporal constraints of the form (15). If there exists a non trivial solution Y to the following inequality:

$$\begin{pmatrix} B_{z_1'} \\ \cdots \\ B_{z_Z'} \end{pmatrix} \otimes Y \leq \begin{pmatrix} \tau_{z_1}^{\max} & & \varepsilon \\ & \cdots & \\ \varepsilon & & \tau_{z_Z}^{\max} \end{pmatrix} \begin{pmatrix} B_{z_1} \\ \cdots \\ B_{z_Z} \end{pmatrix} \otimes Y \tag{22}$$

then the Z constraints are met using the causal control law $u(k) = \bigoplus_{z=1}^{Z} u^z(k)$, where $u^z(k)$ is the control law given by Lemma 2, that guarantees that the zth constraint is satisfied.

Proof: A control law $u^z(k)$ satisfies the zth constraint if and only if Lemma 2 holds. Thus, in order to satisfy condition of Lemma 2 for each $z = 1$ to Z constraints, one has to determine a feedback F that satisfy $\bigoplus_{\ell=1}^{q} B_{z_s'\ell} \otimes F_{\ell r} \leq \bigoplus_{\ell=1}^{q} \tau_{z_s}^{\max} \otimes B_{z_s l} \otimes F_{\ell r}$ for $r = 1$ to N, and for all $s = 1$ to Z. These Z conditions are summarized in inequality (22). The set of solutions to such an inequality is given by an

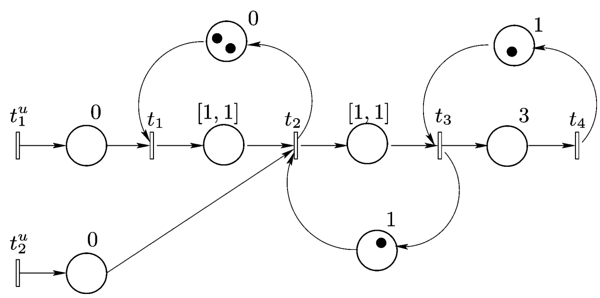

Fig. 3. TEG with temporal constraints on p_{21} and p_{32}.

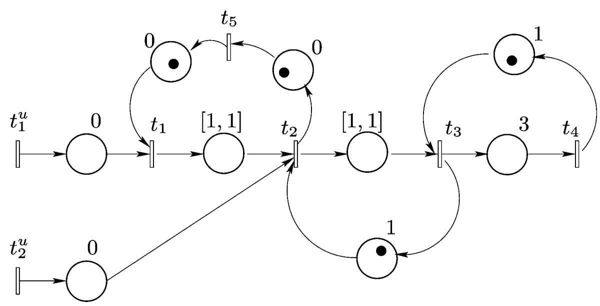

Fig. 4. TEG of Fig. 3 with additional transition t_5.

iterative procedure, as in Theorem 1. If the set is empty, no solution can be found to satisfy all Z constraints together. When a solution exists, a suitable feedback F should also be a solution of inequality (22). ■

Notice that if $Z = 1$, inequality (22) reduces to the condition of Lemma 2, which can be verified using Lemma 1.

VI. Example

Let us consider the TEG of Fig. 3. This graph contains two source transitions modelling respectively $u_1(k)$ and $u_2(k)$ controls, $q = 2$.

Two additional temporal constraints $\{\tau_{21}^{\max}, \tau_{32}^{\max}\}$ are assigned to places p_{21} and p_{32} of this graph, and they are expressed respectively by the inequalities $x_2(k) \leq 1 \otimes x_1(k)$, and $x_3(k) \leq 1 \otimes x_2(k)$. The problem consists in calculating a control vector $u(k) = (u_1(k)u_2(k))^t$, which satisfies both constraints. The graph of Fig. 3 has been transformed into the graph of Fig. 4. To do so, place p_{12} containing 2 tokens has been split into two places marked to 1 and the intermediate transition t_5 has been added.

The state equation associated with this new TEG is

$$x(k) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & 1 & \varepsilon & 1 \\ \varepsilon & \varepsilon & 2 & 1 & 2 \\ \varepsilon & \varepsilon & 5 & 4 & 5 \\ \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \otimes x(k-1) \oplus \begin{pmatrix} e & \varepsilon \\ 1 & e \\ 2 & 1 \\ 5 & 4 \\ \varepsilon & \varepsilon \end{pmatrix} \otimes u(k)$$

where the components of $x(k)$ are the firing times of transitions t_1, t_2, t_3, t_4 and t_5, and the vector $u(k)$ is the control law. The meeting of constraints $\tau_{21}^{\max} = 1$ and $\tau_{32}^{\max} = 1$ is obtained applying Lemma 2 and Theorem 4.

The initial marking of place p_{21} is $m_{21} = 0$. There exists a path α_1 from transition t_1^u to transition t_1 and its initial marking is $m_{\alpha_1} = 0$. One can check that $\tau_{21}^{\max} \otimes B_{11} \geq B_{21}$, condition of Lemma 2, is satisfied. The initial marking of place p_{32} is $m_{32} = 0$. There exists a path α_2 from transition t_2^u to transition t_2 and its initial marking is $m_{\alpha_2} = 0$. One can check that $\tau_{32}^{\max} \otimes B_{2\ell} \geq B_{3\ell}$, $\ell \in \{1, 2\}$, i.e., condition of Lemma 2 is again satisfied. Furthermore, choosing $F_{\ell r} = \varepsilon \, \forall \ell, r \in N$ except $F_{13} = F_{14} = F_{15} = F_{24} = 0$, results in a suitable feedback

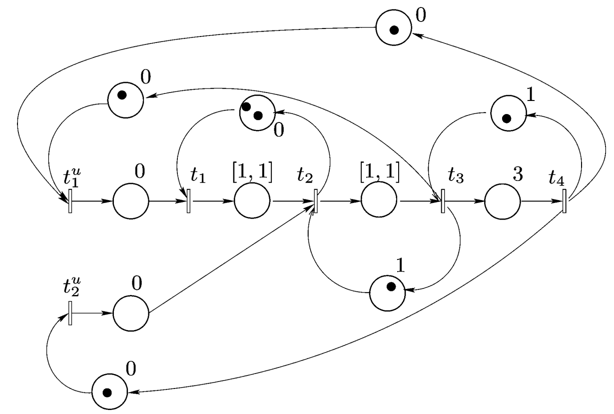

Fig. 5. Resulting controlled TEG: temporal constraints on p_{21} and p_{32} are satisfied by places added between transitions t_3 and t_1^u, t_4 and t_1^u, and transitions t_4 and t_2^u.

that satisfy $\tau_{21}^{\max} \otimes \bigoplus_{\ell=1}^{2} (B_{1\ell} \otimes F_{\ell r}) \geq A_{2r} \oplus \bigoplus_{\ell=1}^{2} (B_{2\ell} \otimes F_{\ell r})$, for $r = 1$ to 5, and $\tau_{32}^{\max} \otimes \bigoplus_{\ell=1}^{2} (B_{2\ell} \otimes F_{\ell r}) \geq A_{3r} \oplus \bigoplus_{\ell=1}^{2} (B_{3\ell} \otimes F_{\ell r})$,for $r = 1$ to 5. Finally, according to Theorem 4, the control law which guarantees the satisfaction of both temporal constraints is given by $u(k) = \begin{pmatrix} x_3(k-1) \oplus x_4(k-1) \oplus x_5(k-1) \\ x_4(k-1) \end{pmatrix}$, that can be reduced to $u(k) = \begin{pmatrix} x_3(k-1) \oplus x_4(k-1) \\ x_4(k-1) \end{pmatrix}$. This feedback can be interpreted by three control places connected to the TEG to guarantee the respect of the temporal constraints. The controlled graph is given in Fig. 5.

VII. Conclusion

We have proposed sufficient conditions for the existence of a control for TEGs subject to strict temporal constraints. The results include the case multivariable control. This method is illustrated on a simple example from manufacturing. The method is explicitely constructive, provided that the conditions hold, leading to a simple control design.

The presented approach in the Max-Plus algebra resembles to the one in the Min-Plus algebra already developed in [4]. We point out that the conditions obtained in Max-Plus algebra are simpler than that obtained in the Min-Plus case. Just one inequality has to be checked in Theorem 2, which is a very simple test to the existence of a solution. In addition, the Max-Plus control law is easier to compute, since there are no need to compute matrix exponent as in the Min-Plus case. Furthermore, depending on the plant to be studied, it can be a great advantage to deal with the Max-Plus approach, especially if the delays involved in the model lay within a wide range of values: this could lead to a very large state vector in the Min-Plus case.

One can note that both Min-Plus and Max-Plus approaches lead to a supervisor that, if the conditions are fulfilled, guarantees that the time constraints are satisfied. For a system with an empty path $(m_\alpha = 0)$ from the control transition t^u to the input transition t_j of the constrained place, a trivial control law $u(k) = +\infty$, for $k \in \mathbb{N}$ guarantees that the time constraints are satisfied. In a production plant context, this trivial control is unacceptable, and one searches for a control law that does not unnecessarily slow down the manufacturing plant throughput. The proposed approach gives a minimal supervisor, which in general is not infimal, since an infimal supervisor does not always exist. This contrasts the work of [11], [14], [18]–[20], where supremal supervisors are proposed, to solve various control problems, that do refer to Just-In-Time optimality criteria.

It was noticed in the introduction that an alternative formulation for this family of problems is in terms of (A, B)-invariance. This fact was pointed out by Katz [16], and generated recent work, see for instance [12]. The solution in this context would come from the computation of a maximal (A, B)-invariant semimodule, which is an open problem in general (see [16]). The present material may help to construct new examples toward a solution to this difficult and important problem.

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